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Parametric interaction of radially gaussian modes in the absence of self-focusing

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Abstract. This paper presents an investigation on the effect of transverse non-uniformity on the parametric interaction of radially gaussian modes of arbitrary type in the absence of self-focusing. The propagation-distance dependences of the real-valued axial amplitudes, phases, beamwidth parameters and phase gradients have been derived by using the variational technique. The analysis has been numerically illustrated by considering the stimulated Raman scattering in a plasma. It is seen that the pump mode propagates like a trapped filament whereas the remaining modes undergo aperiodic focusing or defocusing on account of the parametric-interaction-induced energy exchange. In a particular case, in which the pump mode is much narrower than the remaining modes, all the modes are seen to propagate like trapped filaments.

1. Introduction

The phenomenon of parametric excitation (Louisell 1962, Bloembergen 1968, Yariv 1975) has a long and interesting history. Recently some investigations (Sodha *et al* 1975, Patel 1978) have dealt with the parametric interaction of radially gaussian modes. However, they are primarily concerned with the effect of self-focusing (Shen 1975, Marburger 1975, Sodha *et al* 1978), which occurs when the dielectric constant depends on the intensity. To the best of the authors' knowledge, so far there has been no systematic investigation on the effect of transverse non-uniformity on the parametric interaction in the absence of self-focusing. The present investigation deals with the parametric interaction of radially gaussian modes in the absence of self-focusing and is concerned with the plausibility of focusing–defocusing of modes due to the parametric-interaction-induced energy exchange.

Section 2 opens with the wave equations describing the parametric interaction of three modes of arbitrary type. The equations are based on the assumptions of coaxial propagation, monochromaticity and frequency-matching of the modes. Assuming the modes to have radially gaussian intensity-profiles, the variational technique (Anderson and Bonnedal 1979) has been employed to simplify the coupled partial differential equations for the complex-valued amplitudes. The resulting equations are twelve coupled second-order ordinary differential equations for the real-valued axial amplitudes, phases, beamwidth parameters and phase gradients of the modes. Section 3 simplifies the foregoing analysis by making the additional assumption of wavenumber-matching of the modes. The resulting equations are twelve coupled first-order ordinary differential equations. Section 4 presents numerical results corresponding to the phenomenon of stimulated Raman scattering in a plasma (Forslund *et al* 1975, Thompson and Simon 1976).

It is seen that the real-valued axial amplitudes of the modes vary periodically with the propagation distance and indicate no influence of the transverse non-uniformity of the modes. The beamwidth parameter of the pump mode does not undergo any variation with the propagation distance; however, the beamwidth parameters of the remaining (i.e. signal and idler) modes undergo a significant amount of aperiodic focusing or defocusing. The variations of the phase gradients of the modes are seen to be closely related to the variations of the beamwidth parameters. In a particular case in which the pump mode is much narrower than the remaining modes, all the modes are seen to propagate like trapped filaments.

2. Wavenumbers not matched

It will be assumed that the modes undergoing parametric interaction propagate along the positive z axis and that all the modes have axisymmetric beam-profiles. Thus z will denote the distance of propagation and $r=(x^2+y^2)^{1/2}$ will denote the radial coordinate of the cylindrical coordinate system. The temporal variations of the modes will be assumed to be strictly sinusoidal. The complex-valued time-dependent amplitudes $\tilde{\Psi}_j(t, r, z)$ (where $j=1, 2, 3$) are then expressed in the form

$$\tilde{\Psi}_j(t, r, z)=\Psi_j(r, z) \exp(-\mathrm{i}\omega_j t). \tag{2.1}$$

The modes will be assumed to be temporally resonant so that the frequency-matching condition

$$\sum \omega_j=\omega_1+\omega_2+\omega_3=0 \tag{2.2}$$

is satisfied.

In contrast to the earlier investigations (Sodha *et al* 1975, Patel 1978), the present investigation is aimed at checking whether the transversely non-uniform modes can be focused/defocused solely on account of the parametric-interaction-induced energy exchange. In view of this, it will be assumed that there is no intensity-dependent (Shen 1975, Marburger 1975, Sodha *et al* 1978) variation of the medium which may lead to intensity-coupling of the modes. Under these conditions, the time-independent complex-valued amplitudes $\Psi_j(r, z)$ are governed by the following (Louisell 1962, Bloembergen 1968, Yariv 1975) set of coupled partial differential equations:

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}+k_1^2\right)\Psi_1=M_1\Psi_2^*\Psi_3^* \tag{2.3a}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}+k_2^2\right)\Psi_2=M_2\Psi_3^*\Psi_1^* \tag{2.3b}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}+k_3^2\right)\Psi_3=M_3\Psi_1^*\Psi_2^*. \tag{2.3c}$$

The 'dispersion-characterising quantities' k_j^2 (where $j=1, 2, 3$) and the coupling coefficients M_j are real-valued constants which do not depend on the intensity $|\Psi|^2$.

Except in a few simple cases, equations (2.3) cannot be solved unless some form of the solutions $\Psi_j(r, z)$ is presumed beforehand. If the modes are initially radially gaussian and do not undergo any drastic modifications in the course of propagation, then the solutions of equations (2.3) may be presumed to be of the form

$$\Psi_j=\chi_j \exp(\mathrm{i}\theta_j-\lambda_j r^2+\mathrm{i}\varphi_j r^2). \tag{2.4}$$

The axial amplitudes χ_j, phases θ_j', beamwidth parameters λ_j and phase gradients φ_j are z-dependent real quantities. The exponent appearing in equation (2.4) does not involve terms containing r^4, r^6 etc. This is an approximation which simplifies the analysis to a great extent and is reasonable in the 'macroscopic-average' sense (Marburger and Dawes 1968).

Equations (2.3) can be solved by the Akhmanov approach, which is well known in the literature on self-focusing (Shen 1975, Marburger 1975, Sodha *et al* 1978). However, the Akhmanov approach is of very limited validity. Recently, a better approach (Anderson and Bonnedal 1979) based on the variational principle (Fox 1960) has been adopted in studying the self-focusing phenomenon. The variational technique fits the present problem. The technique is as follows: equations (2.3) are expressed as the Euler–Lagrange equations

$$\frac{\partial}{\partial r}\frac{\partial L}{\partial(\partial\Psi_j^*/\partial r)}+\frac{\partial}{\partial z}\frac{\partial L}{\partial(\partial\Psi_j^*/\partial z)}=\frac{\partial L}{\partial\Psi_j^*}. \tag{2.5}$$

The Lagrangian

$$L(\partial\Psi_j^*/\partial r,\ \partial\Psi_j/\partial r,\ \partial\Psi_j^*/\partial z,\ \partial\Psi_j/\partial z,\ \Psi_j^*,\ \Psi_j; j=1, 2, 3)$$

is integrated to obtain

$$\mathscr{L}(\chi_j, \theta_j, \lambda_j, \varphi_j; j=1, 2, 3)=\int_0^\infty L\,\mathrm{d}r. \tag{2.6}$$

The integrated Lagrangian obeys the new Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial\mathscr{L}}{\partial(\mathrm{d}\alpha_j/\mathrm{d}z)}=\frac{\partial\mathscr{L}}{\partial\alpha_j} \tag{2.7}$$

where $j=1, 2, 3$ and α stands for χ, θ, λ, φ.

This variational technique (Anderson and Bonnedal 1979) leads to the following twelve coupled second-order ordinary differential equations for α_j (where $j=1, 2, 3$):

$$\frac{\mathrm{d}^2\chi_j}{\mathrm{d}z^2}=\frac{4M_j\chi\lambda_j^2}{\chi_j(\lambda^2+\varphi^2)^2}\left[\left(\frac{\lambda}{\lambda_j}(\lambda^2+\varphi^2)-(\lambda^2-\varphi^2)\right)C-\left(\frac{\lambda^2+\varphi^2}{\lambda_j}-z\lambda\right)\varphi S\right]$$
$$+\frac{\chi_j}{2\lambda_j^2}\left[\left(\frac{\mathrm{d}\lambda_j}{\mathrm{d}z}\right)^2-\left(\frac{\mathrm{d}\varphi_j}{\mathrm{d}z}\right)^2\right]+\chi_j\left[\left(\frac{\mathrm{d}\theta_j}{\mathrm{d}z}\right)^2-k_j^2+4\lambda_j\right] \tag{2.8}$$

$$\frac{\mathrm{d}^2\theta_j}{\mathrm{d}z^2}=-\frac{4M_j\chi\lambda_j^2}{\chi_j^2(\lambda^2+\varphi^2)^2}\left[\left(\frac{\lambda^2+\varphi^2}{\lambda_j}-2\lambda\right)\varphi C+\left(\frac{\lambda}{\lambda_j}(\lambda^2+\varphi^2)-(\lambda^2-\varphi^2)\right)S\right]$$
$$-\frac{1}{\lambda_j^2}\frac{\mathrm{d}\lambda_j}{\mathrm{d}z}\frac{\mathrm{d}\varphi_j}{\mathrm{d}z}-\frac{2}{\chi_j}\frac{\mathrm{d}\chi_j}{\mathrm{d}z}\frac{\mathrm{d}\theta_j}{\mathrm{d}z}-4\varphi_j \tag{2.9}$$

$$\frac{\mathrm{d}^2\lambda_j}{\mathrm{d}z^2}=\frac{4M_j\chi\lambda_j^3}{\chi_j^2(\lambda^2+\varphi^2)^2}\left[\left(\frac{\lambda}{\lambda_j}(\lambda^2+\varphi^2)-2(\lambda^2-\varphi^2)\right)C-\left(\frac{\lambda^2+\varphi^2}{\lambda_j}-4\lambda\right)\varphi S\right]$$
$$+\frac{2}{\lambda_j}\left[\left(\frac{\mathrm{d}\lambda_j}{\mathrm{d}z}\right)^2-\left(\frac{\mathrm{d}\varphi_j}{\mathrm{d}z}\right)^2\right]+4(\lambda_j^2-\varphi_j^2)-\frac{2}{\chi_j}\frac{\mathrm{d}\chi_j}{\mathrm{d}z}\frac{\mathrm{d}\lambda_j}{\mathrm{d}z}-2\frac{\mathrm{d}\theta_j}{\mathrm{d}z}\frac{\mathrm{d}\varphi_j}{\mathrm{d}z} \tag{2.10}$$

$$\frac{d^2\varphi_j}{dz^2}=\frac{4M_j\chi\lambda_j^3}{\chi_j^2(\lambda^2+\varphi^2)^2}\left[\left(\frac{\lambda^2+\varphi^2}{\lambda_j}-4\lambda\right)\varphi C+\left(\frac{\lambda}{\lambda_j}(\lambda^2+\varphi^2)-2(\lambda^2-\varphi^2)\right)S\right]$$
$$-\frac{2}{\chi_j}\frac{d\chi_j}{dz}\frac{d\varphi_j}{dz}+\frac{4}{\lambda_j}\frac{d\lambda_j}{dz}\frac{d\varphi_j}{dz}+2\frac{d\theta_j}{dz}\frac{d\lambda_j}{dz}+8\lambda_j\varphi_j. \tag{2.11}$$

The following notations have been used here:

$$(\chi,\theta,\lambda,\varphi)\equiv(\chi_1\chi_2\chi_3,\ \theta_1+\theta_2+\theta_3,\ \lambda_1+\lambda_2+\lambda_3,\ \varphi_1+\varphi_2+\varphi_3) \tag{2.12}$$

$$(C,S)\equiv(\cos\theta,\ \sin\theta). \tag{2.13}$$

Equations (2.8)–(2.11) can be numerically solved (Scarborough 1966, Davis and Polonsky 1972) by using the Runge–Kutta method. As boundary conditions, the values of $\alpha_j(z=0)$ and $(d\alpha_j/dz)_{(z=0)}$ ought to be given beforehand. Whereas the values of $\alpha_j(z=0)$ are defined by the expressions given for $\Psi_j(r,z=0)$, the values of the derivatives $(d\alpha_j/dz)_{(z=0)}$ have to be evaluated in accordance with the physics of the problem. If

$$\theta_j(z=0)=\varphi_j(z=0)=0 \tag{2.14}$$

it is reasonable to assume that

$$(d\alpha_j/dz)_{(z=0)}=\delta_{\alpha\theta}k_j. \tag{2.15}$$

3. Wavenumbers matched

Generally a wave oscillates sinusoidally with the propagation distance (Whitham 1974). This kind of propagation is expressed by a factor $\exp(ikz-i\omega t)$, where the wavenumber k is related to the frequency ω through a dispersion relation. Accordingly, the complex-valued amplitudes $\bar{\Psi}_j(t,r,z)$ (where $j=1,2,3$) may be expressed as

$$\bar{\Psi}_j(t,r,z)=\Psi_j(r,z)\exp(ik_jz-i\omega_jt). \tag{3.1}$$

Note that $\Psi_j(r,z)$ of the present section is $\exp(-ik_jz)$ times the $\Psi_j(r,z)$ of the previous section. Since Ψ_j vary slowly with z as compared to $\exp(ik_jz)$, the WKBJ approximation (Shen 1975, Marburger 1975, Sodha *et al* 1978) may be used to neglect $\partial^2\Psi_j/\partial z^2$ as compared to $2ik_j\partial\Psi_j/\partial z$. In the present section, it will be assumed that the modes are spatially as well as temporally resonant. Thus they satisfy the wavenumber-matching condition (Louisell 1962, Bloembergen 1968, Yariv 1975)

$$\sum k_j=k_1+k_2+k_3=0. \tag{3.2}$$

Then equations (2.3) reduce into the following (Louisell 1962, Bloembergen 1968, Yariv 1975) equations:

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+2ik_1\frac{\partial}{\partial z}\right)\Psi_1=M_1\Psi_2^*\Psi_3^* \tag{3.3a}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+2ik_2\frac{\partial}{\partial z}\right)\Psi_2=M_2\Psi_3^*\Psi_1^* \tag{3.3b}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+2ik_3\frac{\partial}{\partial z}\right)\Psi_3=M_3\Psi_1^*\Psi_2^*. \tag{3.3c}$$

The solutions of equations (3.3) are expressed in the form of equation (2.4). The variational technique (Anderson and Bonnedal 1979) mentioned earlier leads to the following twelve coupled first-order ordinary differential equations for χ_j, θ_j, λ_j and φ_j (where $j=1, 2, 3$):

$$\frac{d\chi_j}{dz}=-\frac{2M_j\chi\lambda_j^2}{k_j\chi_j(\lambda^2+\varphi^2)^2}\left[\left(\frac{\lambda^2+\varphi^2}{\lambda_j}-2\lambda\right)\varphi C+\left(\frac{\lambda}{\lambda_j}(\lambda^2+\varphi^2)-(\lambda^2-\varphi^2)\right)S\right]-\frac{2\chi_j\varphi_j}{k_j} \quad (3.4)$$

$$\frac{d\theta_j}{dz}=\frac{2M_j\chi\lambda_j^2}{k_j\chi_j^2(\lambda^2+\varphi^2)^2}\left[-\left(\frac{\lambda}{\lambda_j}(\lambda^2+\varphi^2)-(\lambda^2-\varphi^2)\right)C+\left(\frac{\lambda^2+\varphi^2}{\lambda_j}-2\lambda\right)\varphi S\right]-\frac{2\lambda_j}{k_j} \quad (3.5)$$

$$\frac{d\lambda_j}{dz}=-\frac{2M_j\chi\lambda_j^3}{k_j\chi_j^2(\lambda^2+\varphi^2)^2}\left[\left(\frac{\lambda^2+\varphi^2}{\lambda_j}-4\lambda\right)\varphi C+\left(\frac{\lambda}{\lambda_j}(\lambda^2+\varphi^2)-2(\lambda^2-\varphi^2)\right)S\right]-\frac{4\lambda_j\varphi_j}{k_j} \quad (3.6)$$

$$\frac{d\varphi_j}{dz}=\frac{2M_j\chi\lambda_j^3}{k_j\chi_j^2(\lambda^2+\varphi^2)^2}\left[\left(\frac{\lambda}{\lambda_j}(\lambda^2+\varphi^2)-2(\lambda^2-\varphi^2)\right)C-\left(\frac{\lambda^2+\varphi^2}{\lambda_j}-4\lambda\right)\varphi S\right]$$
$$+\frac{2}{k_j}(\lambda_j^2-\varphi_j^2). \quad (3.7)$$

The symbols χ, λ, φ, C and S are defined by equations (2.12)–(2.13).

Equations (3.4)–(3.7) may be obtained from equations (2.8)–(2.11) by the following substitutions:

$$\left(\frac{d\theta_j}{dz}\right)^2=k_j^2+2k_j\left(\frac{d\theta_j}{dz}\right) \quad (3.8)$$

$$\left(\frac{d\theta_j}{dz}\right)\left(\frac{d\sigma_j}{dz}\right)=k_j\left(\frac{d\sigma_j}{dz}\right) \quad (3.9)$$

$$\left(\frac{d^2\sigma_j}{dz^2}\right)=\left(\frac{d^2\theta_j}{dz^2}\right)=\left(\frac{d\sigma_j}{dz}\right)^2=0 \quad (3.10)$$

where σ stands for χ, λ and φ. Equations (3.4)–(3.7) can be numerically solved (Scarborough 1966, Davis and Polonsky 1972) by using the Runge Kutta method followed by the Adam method. As boundary conditions, it is required to know the values of $\alpha_j(z=0)$ but not of $(d\alpha_j/dz)_{(z=0)}$.

4. Stimulated Raman scattering as an illustration

The phenomenon of stimulated Raman scattering in a fully ionised plasma (Forslund *et al* 1975, Thompson and Simon 1976) means the conversion of an electromagnetic wave into another electromagnetic wave and a plasma wave. The 'modal' amplitudes involved in this phenomenon are the electric fields E_1 and E_2 of electromagnetic waves at frequencies ω_1 and ω_2 respectively, and the component n of the electron concentration oscillating at frequency ω_3. The wavenumbers k_1, k_2 and k_3 corresponding to these three modes are given by

$$k_1^2=(\omega_1^2-\omega_p^2)c^{-2} \quad (4.1a)$$

$$k_2^2=(\omega_2^2-\omega_p^2)c^{-2} \quad (4.1b)$$

$$k_3^2=(\omega_3^2-\omega_p^2)s^{-2} \quad (4.1c)$$

where the plasma frequency ω_p and the sound speed s are defined by $\omega_p^2=4\pi Ne^2/m$

and $s^2 = K_BT/m$, where N is the background electron concentration and T is the electron temperature.

The process of stimulated Raman scattering in a plasma is significant only in the region where the frequency-matching condition, equation (2.2), and the wavenumber-matching condition, equation (3.2), are satisfied. The equations of motion and continuity combined with the Maxwell equations then lead to the following (Forslund *et al* 1975, Thompson and Simon 1976) coupled partial differential equations for E_1, E_2 and n:

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+2ik_1\frac{\partial}{\partial z}\right)E_1=\left(\frac{2\pi e^2\omega_1}{mc^2\omega_2}\right)n^*E_2^* \tag{4.2a}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+2ik_2\frac{\partial}{\partial z}\right)E_2=\left(\frac{2\pi e^2\omega_2}{mc^2\omega_1}\right)n^*E_1^* \tag{4.2b}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+2ik_3\frac{\partial}{\partial z}\right)n=\left(\frac{Ne^2k_3^2}{2m^2\omega_1\omega_2}\right)E_1^*E_2^*. \tag{4.2c}$$

Equations (4.2) may be written as equations (3.3) under the following identifications:

$$(\Psi_1, \Psi_2, \Psi_3)=(eE_1/m\omega_1,\ eE_2/m\omega_2,\ n) \tag{4.3}$$

$$(M_1, M_2, M_3)=(2\pi e^2/mc^2,\ 2\pi e^2/mc^2,\ Nk_3^2/2s^2). \tag{4.4}$$

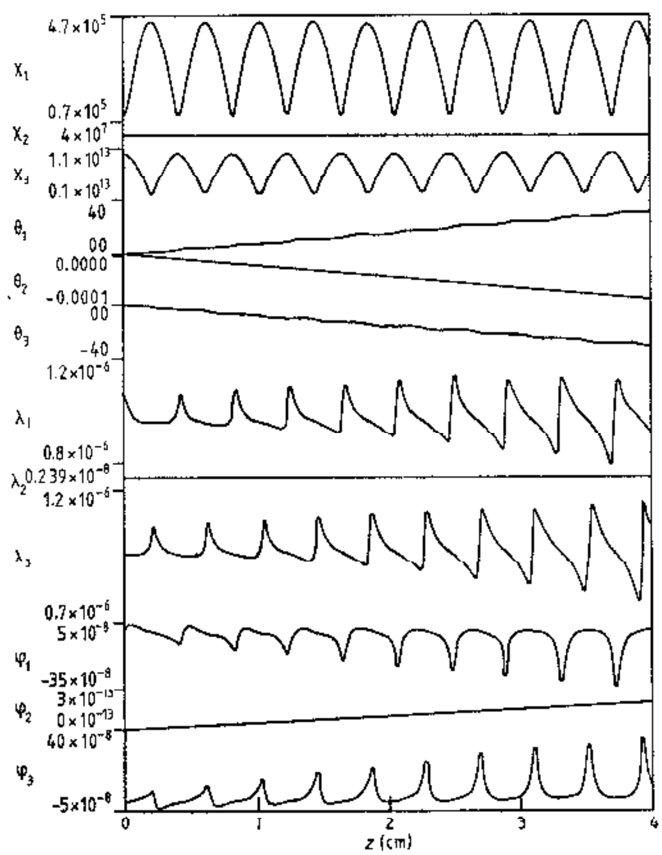

Figure 1. Mode behaviour in the case of very wide beams: $k_j^2/\lambda_{j(0)}=10^{10}$. For other particulars, see the text.

For the purpose of a numerical illustration, the following parameters have been considered:

$$\chi_{1(0)} = 10^5 \text{ cm s}^{-1} \tag{4.5a}$$

$$\chi_{2(0)} = 4 \times 10^7 \text{ cm s}^{-1} \tag{4.5b}$$

$$\chi_{3(0)} = 10^{13} \text{ cm}^{-3} \tag{4.5c}$$

$$\theta_{1(0)} = \theta_{2(0)} = \theta_{3(0)} = 0 \tag{4.6}$$

$$\frac{\lambda_{1(0)}}{k_1^2} = \frac{\lambda_{2(0)}}{k_2^2} = \frac{\lambda_{3(0)}}{k_3^2} = 10^{-10} \qquad \text{for figure 1} \tag{4.7a}$$

$$\frac{\lambda_{1(0)}}{k_1^2} = \frac{\lambda_{2(0)}}{k_2^2} = \frac{\lambda_{3(0)}}{k_3^2} = \frac{1}{1600} \qquad \text{for figure 2} \tag{4.7b}$$

$$10^4 \frac{\lambda_{1(0)}}{k_1^2} = 10^8 \frac{\lambda_{2(0)}}{k_2^2} = 10^4 \frac{\lambda_{3(0)}}{k_3^2} = 1 \qquad \text{for figure 3} \tag{4.7c}$$

$$10^{10} \frac{\lambda_{1(0)}}{k_1^2} = 1600 \frac{\lambda_{2(0)}}{k_2^2} = 10^{10} \frac{\lambda_{3(0)}}{k_3^2} = 1 \qquad \text{for figure 4} \tag{4.7d}$$

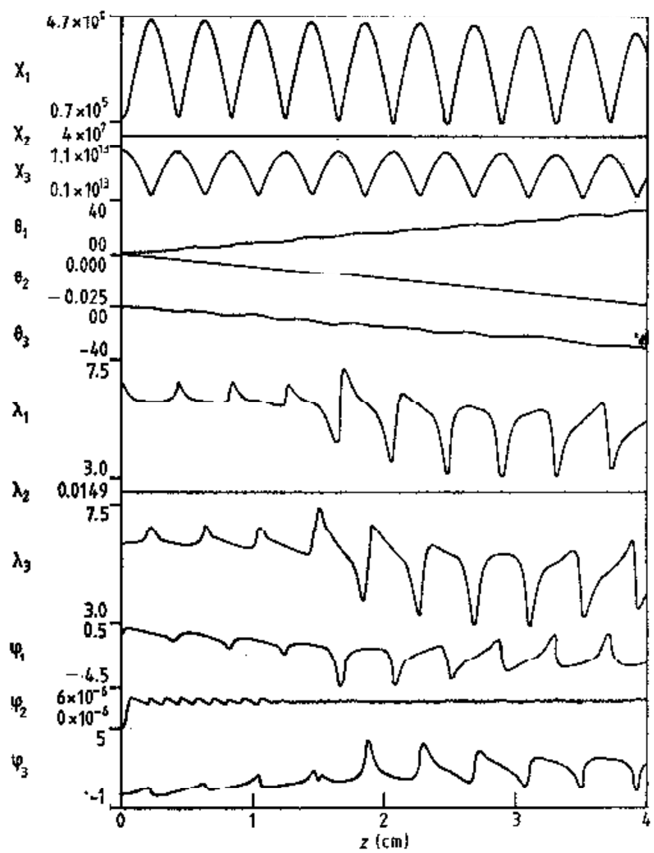

Figure 2. Mode behaviour in the case of very narrow beams: $k_j^2/\lambda_{j(0)} = 1600$.

$$\varphi_{1(0)} = \varphi_{2(0)} = \varphi_{3(0)} = 0 \tag{4.8}$$

$$\omega_1 = -3.574 \times 10^{12}\ \mathrm{s}^{-1} \tag{4.9a}$$

$$\omega_2 = 1.790 \times 10^{12}\ \mathrm{s}^{-1} \tag{4.9b}$$

$$\omega_3 = 1.784 \times 10^{12}\ \mathrm{s}^{-1} \tag{4.9c}$$

$$N = 10^{15}\ \mathrm{cm}^{-3} \tag{4.10a}$$

$$T = 38\,225\ \mathrm{K}. \tag{4.10b}$$

The subscript (0) has been used to denote the value of the quantity at $z=0$.

Results of the numerical analysis have been presented in the form of graphs in figures 1–4. Figure 1 corresponds to very wide beams, figure 2 to very narrow beams, figure 3 to the pump mode wider than the other two modes, and figure 4 to the pump mode much narrower than the other two modes.

First consider the figures 1–3. The $j=2$ mode is seen to act as the pump mode. Its axial amplitude χ_2 and beamwidth parameter λ_2 do not vary with the propagation distance z; thus, the pump mode propagates like a trapped filament. The phase θ_2 decreases linearly with z. The phase gradient φ_2 increases linearly with z, except in figure 2 where it undergoes aperiodic oscillations.

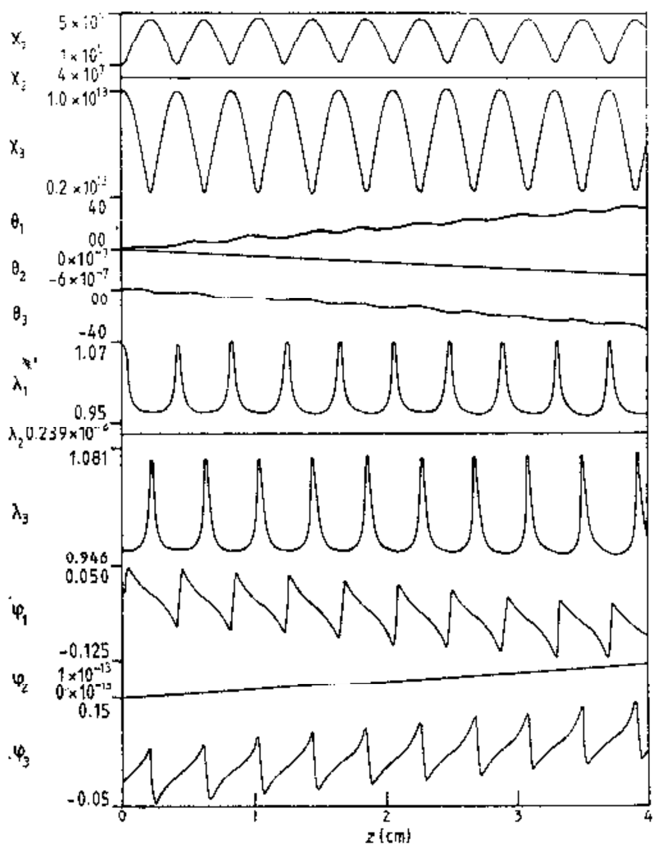

Figure 3. Mode behaviour in the case of the pump mode being wider than the other two modes: $k_2^2/\lambda_{2(0)} = 10^8 = 10^4\, k_{1,3}^2/\lambda_{1,3(0)}$.

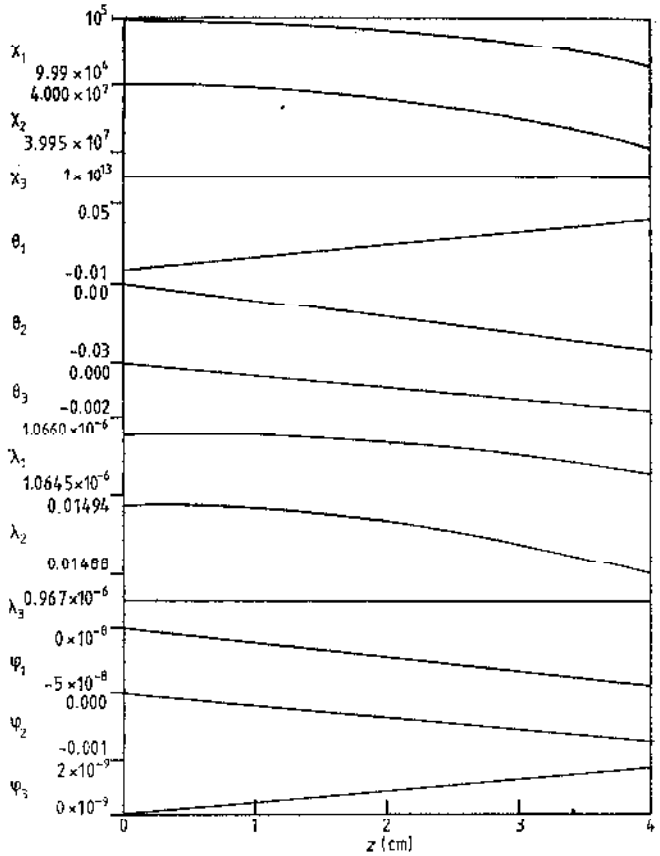

Figure 4. Mode behaviour in the case of the pump mode being much narrower than the other two modes: $k_2^2/\lambda_{2(0)}=1600=1.6\times 10^{-7}\, k_{1,3}^2/\lambda_{1,3(0)}$.

The axial amplitudes χ_1 and χ_3 of the other two modes vary periodically with z and are seen to be unaffected by the transverse non-uniformity of the modes. Because of the opposite signs of their wavenumbers (k_1 and k_3), the $j=1$ and $j=3$ modes behave differently. The axial amplitude χ_1 oscillates between $\chi_{1(0)}$ and $5\chi_{1(0)}$, whereas χ_3 oscillates between $\chi_{3(0)}$ and $\chi_{3(0)}/5$. The phase θ_1 increases, whereas θ_3 decreases; the variations of θ_1 and θ_3 are more pronounced at the points of local minima of χ_1 and χ_3 respectively.

Focusing/defocusing of a mode is manifested by an increase/decrease in the beamwidth parameter λ. The beamwidth parameter λ_1 remains at or below $\lambda_{1(0)}$, whereas λ_3 remains at or above $\lambda_{3(0)}$. Thus the $j=1$ mode tends to get defocused, whereas the $j=3$ mode tends to get focused. At the local minima of χ_1 and χ_3, the beamwidth parameters λ_1 and λ_3 increase sharply, thereby implying sudden focusing. After this sudden focusing, the defocusing of the $j=1$ and $j=3$ modes is gradual, random and sudden in figures 1, 2 and 3 respectively. In this way, the nature of focusing/defocusing of the modes depends very much on whether the modes are initially narrow or wide.

The variations of the phase gradients φ_1 and φ_3 are closely related to the variations of the beamwidth parameters λ_1 and λ_3 respectively. However, unlike the case of conventional self-focusing, it is not possible here to derive an exact relationship between the variations of the beamwidth parameters and phase gradients.

Now consider figure 4. There is a remarkable difference between the curves in figure 4 and the corresponding curves in figures 1–3. Now it is the $j=3$ mode which propagates strictly like a trapped filament. The axial amplitudes χ_1 and χ_2 and the beamwidth parameters λ_1 and λ_2 decrease with z. However, this decrease is insignificant and it is reasonable to say that even the $j=1$ and $j=2$ modes propagate like trapped filaments. In this way, the set of parameters chosen for figure 4 corresponds to the soliton (Whitham 1974, Scott *et al* 1973, Karpman 1975) solution of the problem. The reason for the soliton solution appearing in this particular case is as follows. Since the $j=2$ mode (whose axial intensity far exceeds that of the remaining two modes) is much narrower than the remaining two modes, its off-axial intensity is much less than that of the remaining two modes. Therefore, the $j=2$ mode can donate energy only in the paraxial region; it has to gain energy from the remaining two modes in the off-axial region. As a consequence of such a changeover of the role as the pump mode, there is a radial flow of energy. For the particular case chosen for figure 4, the radial flow of energy happens to nullify the effect of local energy exchange among the three modes.

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