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Temporal growth of a parametric excitation by a self-focused laser beam

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The effect of self-focusing of the pump laser beam on the temporal growth of a parametric excitation has been investigated in the paraxial region. The two equations for the signal and idler modes have been decoupled by assuming the near self-trapping condition and a linearly varying phase mismatch. By employing the WKBJ approximation, it is found that the growth rate is a strong function of the radial intensity inhomogeneity of the pump laser beam. The condition for validity of the first-order approximate theory employed here has been derived.

1. Introduction

The subject of three-wave interaction, commonly known as parametric excitation [1] has been widely investigated. A variety of crystalline and liquid media [2, 3] are now well known for their efficiency in parametric amplification of desired modes. The study of parametric excitation in plasmas has now become indispensable because of its relevance to laser-driven fusion [4]. Inhomogeneity [5], turbulence [6] and boundedness [7] of the medium influence the growth rate of any parametric excitation by varying the phase mismatch. Moreover, inhomogeneity of the medium [8] and non-uniformity of the pump laser beam [9] can influence this growth rate also by varying the coupling coefficient.

A transversely non-uniform laser beam in a non-linear medium undergoes the phenomenon of self-focusing [10–12] and as a result the laser intensity varies with the propagation distance. In this paper, the effect of self-focusing of the pump laser beam on the temporal growth of a parametric excitation has been investigated.

In Section 2, the relevant equations from the theory of self-focusing have been stated and then two dynamic equations for the signal and idler modes of an arbitrary parametric excitation have been set up. These two equations have been decoupled by assuming the near self-trapping condition for the pump laser beam and a linearly varying phase mismatch for the parametric excitation. In Section 3, the single eigenvalue equation obtained upon decoupling has been solved under the WKBJ approximation. An analytic expression has been obtained for the 'growth-rate parameter' from which the temporal growth rate of the parametric excitation can be easily determined. A graph has been plotted to show the variation of this parameter with the 'self-focusing parameter'. In Section 4, a discussion on characteristics of this graph, threshold intensity of the parametric excitation, validity of the assumed first-order approximation and quantization nature of the temporal growth rate has been presented. It is concluded that the temporal growth rate is a strong function of the radial intensity inhomogeneity of the pump laser beam. This strong dependence is not obtained when the pump laser beam is sharply defocused or sharply focused. The results in the cases of back scattering and forward scattering are quite opposite in nature.

The present investigation was carried out before the author noticed two similar investigations [8, 9]. In one [8] of these two, the investigation has not been carried out as thoroughly as the present investigation. In the other one [9], a particular type of parametric excitation has been investigated rigorously, but without illustrating the results by a relevant graph.

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2. Basic equations

The dielectric constant of a non-linear medium irradiated by a moderately intense laser beam is given by [10–12]

$$\epsilon = \epsilon_0 + \epsilon_2 |\mathbf{E}|^2 \tag{1}$$

where ϵ_0 and ϵ_2 depend upon the medium parameters and the laser frequency ω. The electric field of a laser beam, with a radially Gaussian intensity profile and travelling along the z-axis in the medium with the above mentioned dielectric constant is given by

$$\mathbf{E}(r, z, t) = \hat{\mathrm{E}} \frac{E_0}{f(z)} \exp \left\{ -\frac{r^2}{2r_0^2 f^2(z)} + i[\Phi(r, z) + k]z - i\omega t \right\}. \tag{2}$$

Here $k(= \sqrt{\epsilon_0}\,\omega/c)$ is the linear wave number

with

$$\Phi(r, z) \simeq kr^2/[2(z^2 + z_{\mathrm{df}}^2)] + \phi(z) \tag{3}$$

$$z_{\mathrm{df}}^{-2} - z_{\mathrm{d}}^{-2} - z_{\mathrm{f}}^{-2} = 1/(k^2 r_0^4) - \epsilon_2 E_0^2/(2\epsilon_0 r^2) \tag{4}$$

and

$$\phi(z) = [k\epsilon_2 E_0^4/(4\epsilon_0) - 1/(kr_0^2)](z_{\mathrm{df}}/z) \arctan (z/z_{\mathrm{df}})$$

is the non-linear part of the wave number; and

$$f(z) = (1 + z^2/z_{\mathrm{df}}^2)^{1/2} \tag{5}$$

is the beamwidth parameter. The beam gets defocused, trapped or focused depending upon whether $z_{\mathrm{df}}^{-2} >, = \text{or} < 0$.

The expression for $\mathbf{E}(r, z, t)$ (Equation 2) is valid only in the paraxial region ($r \ll r_0$) where Akhmanov *et al.*'s [12] treatment is valid. The expression (Equation 5) for $f(z)$ restricts the validity of the present investigation to a laser beam of not very high intensity and to a small distance of propagation. The dielectric constant saturates [11, 12] as the intensity $|\mathbf{E}|^2$ becomes very large. Consequently, an analytic expression for the beamwidth parameter $f(z)$ is not available in the case of a laser beam of a very high intensity and a large distance of propagation.

Now let the electric vectors associated with the signal and idler modes ($l = 1, 2$) excited by the previously mentioned pump laser beam, in the paraxial region ($r \ll r_0$), be given by

$$\mathbf{E}_l(r \simeq 0, z, t) = \hat{\mathbf{E}}_l E_0 a_l(z) \exp \{i[\phi_l(z) + k_l]z + (g - i\omega_l)t\}. \tag{6}$$

Here the linear wave numbers k_l and frequencies ω_l satisfy the conservation rules [1–3]

$$k_1 + k_2 = k \quad \text{and} \quad \omega_1 + \omega_2 = \omega \tag{7}$$

whereas the non-linear parts $\phi_l(z)$ allow a mismatch

$$\Delta\phi(z) = \phi(z) - [\phi_1(z) + \phi_2(z)]. \tag{8}$$

Due to their coupling with the pump laser beam, the signal and idler modes gain in energy. The gain obviously depends upon the nature of the medium and the parametric excitation under consideration. The gain leads to spatial variation of amplitudes of the signal and idler modes, shift in the frequencies of the two modes, and temporal growth of amplitudes of the two modes. These three consequences are described by the relative amplitude functions $a_l(z)$, imaginary part of the complex function g, and real part of g respectively. The form exp $[\mathrm{Re}\,(g)t]$ for the temporal growth is based on the assumptions that the temporal growths of the two modes are coupled so that they are the same, are exponential so that $[\mathrm{Re}\,(g)t]$ occurs as an exponent, and are uniform so that g is independent of z. These assumptions are quite justified if the pump intensity is not very high so that saturation of the parametric excitation does not come into the picture.

The relative amplitude functions $a_l(z)$ satisfy, in the paraxial region and under the linearization approximation, the coupled rate equations [1, 5, 6]

$$\left(\frac{d}{dz}+\frac{g+\nu_1}{V_1}\right)a_1(z) = \frac{\Gamma_0}{V_1 f(z)}a_2^*(z)\exp\left[i\Delta\phi(z)z\right] \tag{9}$$

$$\left(\frac{d}{dz}+\frac{g+\nu_2}{V_2}\right)a_2^*(z) = \frac{\Gamma_0}{V_2 f(z)}a_1(z)\exp\left[-i\Delta\phi(z)z\right]. \tag{10}$$

Here ν_l and V_l are the damping rates and the group velocities along the z-axis, of the excited modes; and Γ_0 is the 'homogeneous' coupling coefficient which varies linearly with E_0 the proportionality factor depending upon the parametric excitation under consideration. It has been assumed here that the laser-induced spatial dependence of the parameters appearing in the expression for Γ_0 is negligible compared to that of $f(z)$. The linearization approximation employed above is justified when the relative amplitude functions vary slowly in space.

In the first-order approximation, the pump laser beam will be assumed to be approximately self-trapped so that $z_{\mathrm{df}}^{-2} \simeq 0$. Then $f(z)$ is a slowly varying function of z, and hence while decoupling Equations 9 and 10, the term

$$\mu(z) \equiv (d/dz)\ln f(z)$$

may be neglected. Moreover, the mismatch function $\Delta\phi(z)$ will be assumed to be independent of z and set equal to $\Delta\phi_0$. This is reasonable because of the cases of interest, the wave number mismatch is negligible or a slowly/linearly varying function of z. It can now be shown that the transformations [6]

$$a_1(z)\exp\left(-i\Delta\phi_0\frac{z}{2}\right) = a_2^*(z)\exp\left(i\Delta\phi_2\frac{z}{2}\right) = \Psi(z)\exp\left[-\left(\frac{g+\nu_1}{V_1}+\frac{g+\nu_2}{V_2}\right)\frac{z}{2}\right] \tag{11}$$

can reduce Equations 9 and 10 into a single equation

$$\left[\frac{d^2}{dz^2}+W(z)\right]\Psi(z) = 0 \tag{12}$$

where

$$W(z) = \frac{\Gamma_0^2}{V_1\bar{V}_2(1+z^2/z_{\mathrm{df}}^2)}-\left(\frac{g+\nu_1}{2V_1}+\frac{g+\nu_2}{2\bar{V}_2}+\frac{i\Delta\phi_0}{2}\right)^2 \tag{13}$$

and $\bar{V}_2 = -V_2$. For the more common case of back scattering $\bar{V}_2$ is positive; V_1 having been assumed to be positive.

3. Growth rate

Because of the assumed slow z dependence of $f(z)$, $W(z)$ is a slowly varying function of z, and hence a WKBJ solution [13] of Equation 12 is reasonable. Thus

$$\Psi(z) \simeq C_\pm[W(z)]^{-1/4}\exp\left\{\pm i\int\sqrt{[W(z)]}\,dz\right\} \tag{14}$$

where $C_\pm$ are the constant coefficients determined by boundary conditions. The turning points $z = \pm z_t$ where $W(z)$ vanishes are

$$z_t = z_{\mathrm{df}}\left[\left(\frac{g+\nu_1}{2V_1}+\frac{g+\nu_2}{2\bar{V}_2}+\frac{i\Delta\phi_0}{2}\right)^{-2}\frac{\Gamma_0^2}{V_1\bar{V}_2}-1\right]^{1/2}. \tag{15}$$

According to the WKBJ theory [13] of reflection inside a potential well, the eigenvalues g_n satisfy the condition

$$\int_0^{z_t}\sqrt{[W_n(z)]}\,dz = (2n+1)\pi/4 \tag{16}$$

where n is a non-negative integer. This gives

$$B\left[1-\left(\frac{g_n+\nu_1}{2V_1}+\frac{g_n+\nu_2}{2\bar{V}_2}+\frac{i\Delta\phi_0}{2}\right)^2\frac{V_1\bar{V}_2}{\Gamma_0^2}\right] = \frac{(2n+1)^2(\pi/4)^2V_1\bar{V}_2}{\Gamma_0^2 z_{\mathrm{df}}^2} \tag{17}$$

where the notation

$$B(x) = \left\{ \int_0^{\pi/2} [(1 - x \sin^2\theta)^{-1/2} - (1 - x \sin^2\theta)^{1/2}]\, d\theta \right\}^2$$

$$= [\mathcal{K}(x) - \mathcal{E}(x)]^2 \qquad (18)$$

has been used for convenience. Here $\mathcal{K}(x)$ and $\mathcal{E}(x)$ are the well-known complete elliptic functions of the first and second kind [14].

The following analytic expression for g_n, whose real part is the temporal growth rate of the parametric excitation and imaginary part is the frequency shift, then results

$$(g_n\alpha + \beta) = [1 + B^{-1}(\gamma_n z_{\mathrm{df}}^{-2})]^{1/2} \qquad (19)$$

where

$$\alpha = [\sqrt{(\bar{V}_2/V_1)} + \sqrt{(V_1/\bar{V}_2)}]/2\Gamma_0$$

$$\beta = [\nu_1\sqrt{(\bar{V}_2/V_1)} + \nu_2\sqrt{(V_1/\bar{V}_2)} + i\Delta\phi_0\sqrt{(V_1\bar{V}_2)}]/2\Gamma_0$$

$$\gamma_n = -(2n+1)^2(\pi/4)^2 V_1\bar{V}_2/\Gamma_0^2 \qquad (20)$$

and B^{-1} stands for the inverse function corresponding to B. From physical reasoning (by carrying out the foregoing analysis for $z_{\mathrm{df}}^{-2} = -z_{\mathrm{fd}}^{-2}$), it is found that $B^{-1}(-\gamma_n z_{\mathrm{df}}^{-2}) = -B^{-1}(\gamma_n z_{\mathrm{df}}^{-2})$. For convenience, the quantities $(\gamma_n z_{\mathrm{df}}^{-2})$ and $(g_n\alpha + \beta)$ will be termed as 'self-focusing parameter' and 'growth-rate parameter' respectively. In the absence of self-focusing, i.e. when the diffraction effect is balanced by the focusing non-linearity, $\gamma_n z_{\mathrm{df}}^{-2} = 0$ and $(g_n\alpha + \beta) = 1$. The quantity

$$Q = \{100(g_n\alpha + \beta - 1)(1 - \mathrm{Re}\,\beta)/[(1 - \mathrm{Re}\,\beta)^2 + (\mathrm{Im}\,\beta)^2]\}$$

is the percentage increase in the temporal growth rate ($\mathrm{Re}\,g_n$) when the effect of self-focusing is taken into account.

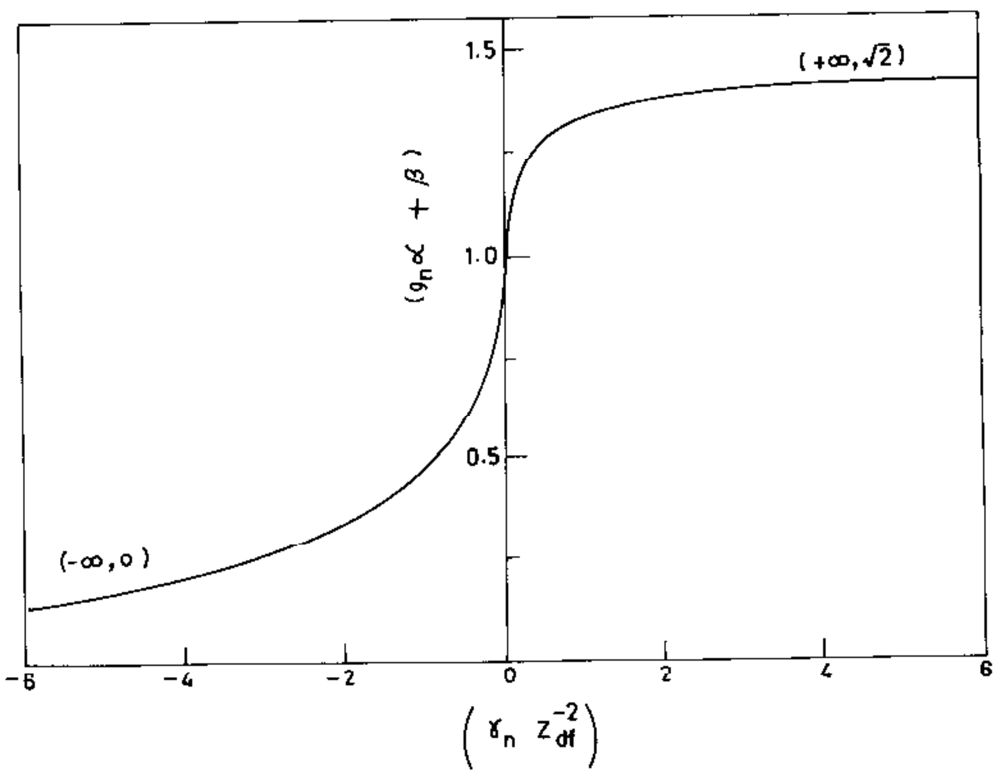

Figure 1 'Growth rate parameter' $(g_n\alpha + \beta)$ versus 'self-focusing parameter' $(\gamma_n z_{\mathrm{df}}^{-2})$.

In Fig. 1, the 'growth-rate parameter' $(g_n\alpha + \beta)$ has been plotted versus the 'self-focusing parameter' $(\gamma_n z_{\rm df}^{-2})$. Note that the graph is separable into two parts, namely upper and lower parts corresponding to $(g_n\alpha + \beta) > 1$ and < 1. The upper (/lower) part corresponds to the case of focusing (/defocusing) and back scattering i.e. $\bar{V}_2 > 0$ or of defocusing (/focusing) and forward scattering i.e. $\bar{V}_2 < 0$.

4. Discussion

In the region $|\gamma_n z_{\rm df}^{-2}| \simeq 0$, where the first-order approximate theory may be quite reasonable, the graph $(g_n\alpha + \beta)$ versus $(\gamma_n z_{\rm df}^{-2})$ has a remarkably large slope. This means that in the near self-trapping condition, the temporal growth rate (Re g_n) varies drastically with the 'self-focusing parameter' $(\gamma_n z_{\rm df}^{-2})$. For example in the case of focusing and back scattering (/defocusing and forward scattering) with $(\gamma_n z_{\rm df}^{-2}) = 0{\cdot}1$, the graph gives $(g_n\alpha + \beta) = 1{\cdot}059$ which implies a 5·9% increase in its conventional value of unity calculated for $(\gamma_n z_{\rm df}^{-2}) = 0$, or equivalently $\{5{\cdot}9(1 - \mathrm{Re}\,\beta)/[(1-\mathrm{Re}\,\beta)^2 + (\mathrm{Im}\,\beta)^2]\}$% increase in the temporal growth rate (Re g_n). Similarly, in the case of defocusing and back scattering (/focusing and forward scattering) with $(\gamma_n z_{\rm df}^{-2}) = -0{\cdot}1$, the graph gives $(g_n\alpha + \beta) = 0{\cdot}937$ which implies a 6·3% decrease in its conventional value, or equivalently $\{6{\cdot}3(1 - \mathrm{Re}\,\beta)/[(1-\mathrm{Re}\,\beta)^2 + (\mathrm{Im}\,\beta)^2]\}$% decrease in the temporal growth rate. Both of these cases thus lead to appreciable changes in the temporal growth rate.

The values ± 0·1 for the 'self-focusing parameter' $(\gamma_n z_{\rm df}^{-2})$ are realizable in a variety of cases. As an example, consider [1] a parametric excitation at frequencies $\omega_1 = \omega_2 = 9{\cdot}4 \times 10^{14}\,\mathrm{s}^{-1}$ by a Nd–glass laser beam of frequency $\omega = 1{\cdot}88 \times 10^{15}\,\mathrm{s}^{-1}$ and of intensity $E_0^2 = 200\,\mathrm{erg\,cm}^{-3}$ in a $LiNbO_3$ crystal for which $\epsilon_0 = 4{\cdot}8$, ϵ_2 (due to thermal changes) $\simeq 10^{-6}\,\mathrm{cm}^3\,\mathrm{erg}^{-1}$ and second-order susceptibility tensor component $d_{15} \simeq 1{\cdot}3 \times 10^{-8}\,\sqrt{(\mathrm{cm}^3\mathrm{erg}^{-1})^{1/2}}$. These parameters give $\gamma_0 \simeq 2{\cdot}5\,\mathrm{cm}^2$. Consequently for $(\gamma_n z_{\rm df}^{-2})$ to become $+\,0{\cdot}1$ and $-\,0{\cdot}1$, the mean beamwidth radius r_0 of the laser beam has to be $1{\cdot}58 \times 10^{-3}$ cm and $1{\cdot}55 \times 10^{-2}$ cm respectively. Laser beams of these r_0 values are commonly available. It is thus concluded that the effect of self-focusing (defocusing or focusing) on the temporal growth rate of the parametric excitation is appreciable in a variety of experimentally realizable situation.

The threshold intensity $I_{\rm th}$ of the parametric excitation under consideration is determined by setting the complex temporal growth-rate function g_n equal to zero in Equation 19. This gives the transcendental equation

$$\frac{(\sqrt{I}\beta)}{\sqrt{I_{\rm th}}} = \left\{1 + B^{-1}\left[\frac{(I\gamma_n)}{I_{\rm th}}\, z_{\rm df}^{-2}\right]\right\}^{1/2}. \tag{21}$$

In the exact self-trapping case, i.e. for $z_{\rm df}^{-2} = 0$, this gives the usual result

$$I_{\rm th0} = (1/4)\,|\nu_1\sqrt{(\bar{V}_2/V_1)} + \nu_2\sqrt{(V_1/\bar{V}_2)} + i\Delta\phi_0\sqrt{(V_1\bar{V}_2)}|^2 \tag{22}$$

which, as expected, is independent of the transverse intensity profile of the laser beam. Equation 21 can be solved for $I_{\rm th}$ by the usual graph cross-over technique. For example, on Fig. 1, $I_{\rm th}$ corresponds to the cross-over point of the already plotted graph of $[1 + B^{-1}(\gamma_n z_{\rm df}^{-2})]^{1/2}$ with a new graph of

$$\{2(i\nu_1/\sqrt{V_1} + i\nu_2/\sqrt{\bar{V}_2} - \Delta\phi_0)\, z_{\rm df}/[(2n+1)\pi]\}(\gamma_n z_{\rm df}^{-2})^{1/2}.$$

It is apparent that in the near self-trapping condition, the threshold intensity varies appreciably with parameters appearing in the expression for $z_{\rm df}^{-2}$.

In Fig. 1, observe that the 'growth-rate parameter' $(g_n\alpha + \beta)$ has upper and lower saturation values $\sqrt{2}$ and 0 as the 'self-focusing parameter' $(\gamma_n z_{\rm df}^{-2})$ approaches $\pm\,\infty$. The upper (/lower) saturation corresponds to the case of complete focusing (/defocusing) and back scattering or complete defocusing (/focusing) and forward scattering. Hence the first-order approximate theory applicable to the near self-trapping region is not expected to remain valid in these saturation regions. In order to get rid of this limitation, the term $\mu(z) \equiv (\mathrm{d}/\mathrm{d}z)\ln f(z)$ should be incorporated in the foregoing theory. With this term included, it is found that Equations 11–13 need to be replaced by

$$\frac{a_1(z)}{\Psi_1(z)}\exp\left[\left(-\mu(z) - i\Delta\phi_0\right)\frac{z}{2}\right] = \frac{a_2^*(z)}{\Psi_2(z)}\exp\left\{\left[-\mu(z) + i\Delta\phi_0\right]\frac{z}{2}\right\} = \exp\left[-\left(\frac{g+\nu_1}{V_1} + \frac{g+\nu_2}{V_2}\right)\frac{z}{2}\right] \tag{23}$$

$$\left(\frac{d^2}{dz^2} + W_l(z)\right)\Psi_l(z) = 0 \tag{24}$$

and

$$W_l(z) = \frac{\Gamma_0^2}{V_1 \bar{V}_2(1 + z^2/z_{df}^2)} - \left[\frac{g+\nu_1}{2V_1} + \frac{g+\nu_2}{2\bar{V}_2} + (-1)^l\frac{\mu(z)}{2} + \frac{i\Delta\phi_0}{2}\right]^2. \tag{25}$$

The WKBJ approximation is still reasonable in the intermediate region where z_{df}^{-2} and $[d\mu(z)/dz]$ are not very large. The foregoing first-order approximate treatment is valid, i.e. the term $\mu(z)$ may be neglected, if and only if

$$\mu(z) \ll \left|\left(\frac{\nu_1}{V_1} + \frac{\nu_2}{\bar{V}_2} + i\Delta\phi_0\right)\right|. \tag{26}$$

This condition can be satisfied when the signal and idler modes are slowly propagating and easily attenuable in the region not far away from the self-trapping of the pump laser beam.

The present investigation is valid in the paraxial region and when $(d/dz)\ln\Gamma_0 \ll z_{df}^{-1}$ so that the spatial variation of Γ_0 may be neglected. The present investigation may be extended to include the non-paraxial region and the effect of spatial variations of Γ_0. However, the analysis would then become much more complicated than the present one and analytic results would not be possible.

Note the important role now played by the quantum number n in the expressions for γ_n and g_n, and also note that this n does not come into the picture for a transversely homogeneous or a self-trapped transversely inhomogeneous laser beam. It should be recalled that g_n is complex so that the 'n_{th} temporal growth rate' corresponds to the parametric excitation at the 'n_{th} frequency shift'.

It is concluded from the present investigation that the temporal growth rate of a parametric excitation varies significantly with the transverse intensity inhomogeneity of the pump laser beam, and hence the effect of self-focusing may not be neglected. The first-order approximation employed here is valid when the signal and idler modes are slowly propagating and easily attenuable in the region not far away from the self-trapping of the pump laser beam.

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