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EXTENSION OF THE PRINCIPLE OF MATHEMATICAL INDUCTION

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Abstract : Principle of mathematical induction has been explored over the sets of all integers, all rational numbers, all real numbers and all complex numbers. This has been done as a natural extension of the known induction principle for the set of all positive integers.

Principle of mathematical induction valid over the set N of all positive integers follows from the N formation principle i.e. Peano's last axiom.

Lemma 1 : Let $K \subset I_a + = \{ a, a+1, a+2, \ldots \}$ where a is some specified integer. $K = I_a+$ if $a \varepsilon K$ and if $k \varepsilon K \longrightarrow k + 1 \varepsilon K$.
Proof is an obvious extension of N formation Principle based on the concept of well-ordering of a set.

Lemma 2 : Let $K \subset I_b- = \{ b, b-1, b-2, \ldots \}$ where b is some specified integer. $K = I_b-$ if $b \varepsilon K$ and if $k \varepsilon K \longrightarrow k - 1 \varepsilon K$.

Proof : I_b- is a down-well-ordered set ; by down-well-ordered set we mean a set each nonempty subset of which contains the greatest element.

Suppose $L = I_b- - K$ is nonempty.

$b \varepsilon I_b-$ and also $b \varepsilon K$. Hence b does not belong to L. L being a subset of a down-well-ordered set, contains some greatest element say g.

obviously $g < b, \longrightarrow (g+1) \leqslant b$. Hence

$(g+1) \varepsilon I_b-$. But $(g+1)$ does not belong to L. Hence $(g+1) \varepsilon K$.

Hence $(g+1) - 1 = g \varepsilon K$.

But this contradicts with $g \varepsilon L$.

This implies that our supposition is wrong, i.e. L cannot be nonempty, i.e. $K = I_b-$.

Theorem 1 : Let $K \subset I$ — set of all integers. $K = I$ if at least one integer say $a \varepsilon K$ and if $k \varepsilon K \longrightarrow k \pm 1 \varepsilon K$.

Corollary 1 : A proposition is true for all integers if it is true for at least one integer and if whenever it is true for some integer k, it is also true for integers $k \pm 1$.

Induction principles being direct consequences of the corresponding formation principles, will nowonwards be omitted.

Corollary 2 : Let $K \subset X = \{ x_i \}$ where $i \varepsilon I$. $K = X$ if for at least one $m \varepsilon I$, $x_m \varepsilon K$ and if $x_k \varepsilon K \longrightarrow x_{k \pm 1} \varepsilon K$.

Theorem 2 : Let $K \subset Q$ = set of all rational numbers. $K = Q$ if at least one rational number say $\frac{d}{d_1} \varepsilon K$ and if

$$\frac{k}{k_1} \varepsilon K \longrightarrow \frac{k \pm 1}{k_1}, \quad \frac{k}{k_1 + 1} \varepsilon K.$$

Where $d, k \varepsilon I$ and $d_1, k_1 \varepsilon N$.

Proof : According to the given conditions, numerator of the rational number $\frac{d}{d_1}$ can be changed to get at least one integer say $e \varepsilon K$.

$$\text{Now } \frac{e}{k_1} \varepsilon K \longrightarrow \frac{e}{k_1 + 1} \varepsilon K.$$

Therefore according to N formation principle, all rational numbers of the form $\frac{e}{m} \varepsilon K$ where $m \varepsilon N$.

$$\text{Now } \frac{k}{m} \varepsilon K \longrightarrow \frac{k \pm 1}{m} \varepsilon K.$$

Therefore according to I formation principle, all rational numbers of the form $\frac{j}{m} \varepsilon K$ where $j \varepsilon I$.

But this means that all the rational numbers εK, i.e. $Q \subset K$. But $K \subset Q$. Therefore $K = Q$.

Theorem 3 : Let $K \subset R$ = set of all real numbers. $K = R$ if at least one real number say $p \varepsilon K$ and if there exists at least one positive real number h such that $k \varepsilon K \longrightarrow$ the open interval $(k - h, k + h) \subset K$.

Proof :—We can take another positive real number $h_1 < h$ so that $k \varepsilon K \longrightarrow$ the closed interval $[k - h_1, k + h_1] \subset K$. $p \varepsilon K$. Hence $[p - h_1, p + h_1] \subset K$.

Hence $p \pm h_1 \varepsilon K$.

$p + h_1 \varepsilon K \longrightarrow [p, p + 2h_1] \subset K$ .. .. $\longrightarrow$

$[p + nh_1 - h_1, p + nh_1 + h_1] \subset K$ where $n \varepsilon N$.

Similarly $p - h_1 \varepsilon K \longrightarrow [p - nh_1 - h_1, p - nh_1 + h_1] \subset K$. All these imply that $R \subset K$. But $K \subset R$. Hence $K = R$.

Theorem 4 : Let $K \subset C$ = set of all complex numbers. $K = C$ if at least one complex number εK and if there exists at least one positive real number h such that $k + ik_1 \varepsilon K \longrightarrow l + ik_1, k + il_1 \varepsilon K$ where $l \varepsilon (k - h, k + h)$ and $l_1 \varepsilon (k_1 - h, k + h)$.

Proof is by lexicographic method as used in theorem 2, and theorem 2, and theorem 3 is to be applied.

Remarks :

The principle can be generalized to any space in which a coordinate frame of appropriate number of dimensions can be adjusted : directly or after stereographic projection. Here we take lexicographic approach by applying corresponding formation principle to each dimensional axis which is isomorphic to N, I or R.

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